HW 11

Page 247 (3a) $\sinh z = (e^z - e^{-z})\frac{1}{2}$

$$= z + \frac{z^3}{6} + \frac{z^5}{120} + \dots$$

$$\frac{\sinh z}{z^4} = \frac{1}{z^3} + \frac{1}{6z} + \dots$$

$m = 3$, $B = 1/6$

(3b) $e^z - 1 = z + \frac{z^2}{2} + \dots$

By long divison,

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \dots$$

$$\frac{1}{z(e^z-1)} = \frac{1}{z^2} - \frac{1}{2z} + \dots$$

$m = 2$, $B = -1/2$.

(5a) $z^3(z+4) = 0 \Rightarrow z = 0$ or $z = -4$

Around $z = 0$, $\frac{1}{z^3(z+4)} = \frac{1}{4z^3}\left(\frac{1}{1-(-z/4)}\right)$

$$= \frac{1}{4z^3}\left(\sum_{n=0}^{\infty}\left(-\frac{z}{4}\right)^n\right) \text{ if } |z| < 4$$

$$\frac{1}{z^3(z+4)} = \sum_{n=-3}^{\infty} (-1)^{n+1} \frac{z^n}{4^{n+4}}$$

Therefore, $\displaystyle\int_C \frac{dz}{z^3(z+4)} = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^3(z+4)}$

$$= \frac{\pi i}{32}$$

(5b) Around $z=-4$,

$$\cancel{\frac{1}{z^3} = \frac{1}{4-4+z^3}} \quad \operatorname*{Res}_{z=-4} \frac{1}{z^3(z+4)} = \lim_{z\to -4} \frac{1}{z^3}$$

$$= \frac{-1}{64}$$

Therefore, $\displaystyle\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(\operatorname*{Res}_{z=0} \frac{1}{z^3(z+4)} + \operatorname*{Res}_{z=-4} \frac{1}{z^3(z+4)} \right)$

$$= 0.$$

(6) Since $\dfrac{\cosh \pi z}{z(z^2+1)}$ has poles at $z=0$, $z=\pm i$ with order 1 for each.

$$\int_C \frac{\cosh \pi z}{z(z^2+1)} = 2\pi i \left(\operatorname*{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} + \operatorname*{Res}_{z=\pm i} \frac{\cosh \pi z}{z(z^2+1)} \right)$$

$$= 2\pi i \left(\lim_{z\to 0} \frac{\cosh \pi z}{z^2+1} + \lim_{z\to i} \frac{\cosh \pi z}{z(z+i)} + \lim_{z\to -i} \frac{\cosh \pi z}{z(z-i)} \right)$$

$= 2\pi i(2) = 4\pi i$

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3a $\sinh z$ and z^2 have no singularity about $z = \pi i/2$. $\frac{1}{\cosh z}$ has pole of order 1 at $z = \frac{\pi i}{2}$

by L'hosptial rules, thus,

$$\operatorname*{Res}_{z=\frac{\pi i}{2}} \frac{\sinh z}{z^2 \cosh z} = \lim_{z \to \frac{\pi i}{2}} \frac{\sinh z\,(z - \pi i/2)}{z^2 \cosh z}$$

$$= \lim_{z \to \frac{\pi i}{2}} \frac{\sinh z + (z - \pi i/2)\cosh z}{2z \cosh z + z^2 \sinh z}$$

$$= -\frac{4}{\pi^2}$$

3b Since $\frac{1}{\sinh z}$ has pole of order 1 at $z = \pi i$

by L'hosptial rule,

$$\operatorname*{Res}_{z=\pi i} \frac{e^{zt}}{\sinh z} = \lim_{z \to \pi i} \frac{e^{zt}(z - \pi i)}{\sinh z}$$

$$= \lim_{z \to \pi i} \frac{e^{zt} + (z - \pi i) t e^{zt}}{\cosh z}$$

$$= -e^{\pi i t}$$

Similarly, $\operatorname*{Res}_{z=-\pi i} \frac{e^{zt}}{\sinh z} = -e^{-\pi i t}$

Therefore, the required ans is $-2\cos(\pi t)$.

4a Since each poles has order 1, thus

$$\operatorname*{Res}_{z=z_n}\left(z\sec z\right) = \lim_{z\to z_n} z\sec z\,(z-z_n)$$

$$= \lim_{z\to z_n} \frac{z(z-z_n)}{\cos z}$$

$$= (-1)^{n+1} z_n.$$

4b ~~Since~~ Similar in **4a**.

5a since the poles $z=\pm\pi/2$ have order 1.

$$\operatorname*{Res}_{z=\pi/2} \tan z = \lim_{z=\pi/2} \frac{(z-\pi/2)\sin z}{\cos z}$$

$$= \lim_{z=\pi/2} \frac{(z-\pi/2)\cos z + \sin z}{-\sin z}$$

$$= -1$$

Thus $$\int_C \tan z\,dz = 2\pi i\left(\operatorname*{Res}_{z=\pi/2} + \operatorname*{Res}_{z=-\pi/2} \tan z\right)$$

$$= -4\pi i$$

(5b) The pole $z=0$, $\pm\pi i/2$ have order 1.

$$\operatorname*{Res}_{z=0}\left(\sinh 2z\right)^{-1} = \lim_{z\to 0}\frac{z}{\sinh 2z}$$

$$= 1/2$$

$$\operatorname*{Res}_{z=\pm\pi i/2}\left(\sinh 2z\right)^{-1} = \lim_{z\to\pm\pi i/2}\frac{\left(z\mp\pi i/2\right)}{\sinh 2z}$$

$$= -1/2$$

$$\int_c \frac{dz}{\sinh 2z} = 2\pi i\left(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z\to\pm\pi i/2}\frac{1}{\sinh 2z}\right)$$

$$= -\pi i.$$

(6) $$\int_{C_N}\frac{dz}{z^2\sin z} = 2\pi i\left(\operatorname*{Res}_{z=0}\frac{1}{z^2\sin z} + \sum_{n=1}^{N}\operatorname*{Res}_{z=\pm n\pi}\frac{1}{z^2\sin z}\right)$$

Around $z=0$, $\frac{1}{z^2\sin z} = \frac{1}{z^2}\left(\frac{1}{z}+\frac{z}{6}+\cdots\right)$

Around $z=\pm n\pi$,

$$\operatorname*{Res}_{z=\pm n\pi}\frac{1}{z^2\sin z} = \lim_{z\to\pm n\pi}\frac{z-\left(\pm n\pi\right)}{z^2\sin z}$$

$$=\left(\frac{1}{n\pi}\right)^2\frac{1}{\cos(n\pi)}$$

$$=\frac{(-1)^n}{n^2\pi^2}$$

Thus $\displaystyle\int_{C_N} \frac{1}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right)$

Since $\displaystyle\int_{C_N} \frac{1}{z^2 \sin z} \to 0$ as $N \to \infty$. Result follows.

(7) Since $\displaystyle \frac{d}{dz} \left\{ (z^2-1)^2 + 3 \right\} \Big|_{z = \frac{\sqrt{3}+i}{2}} \neq 0$.

$$\int_C \frac{dz}{(z^2-1)^2+3} = \left(\operatorname*{Res}_{z=z_0} + \operatorname*{Res}_{z=-\bar{z}_0} \frac{1}{(z^2-1)^2+3} \right) 2\pi i$$

$$= \left(\frac{2\pi i}{4z(z^2-1)} \right) \Big|_{z=z_0} + \left(\frac{2\pi i}{4z(z^2-1)} \right) \Big|_{z=-\bar{z}_0}$$

$$= \frac{\pi}{2\sqrt{2}}$$